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The Discrete Equations of Fluid Flow and Elasticity

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1. INTRODUCTION

We shall operate in Euclidean 3-space, E_3, and deal with the discrete analog of the following system of equations:

$$\begin{aligned} \Delta \mathbf{v} - \nabla p &= -\mathbf{f}, \\ \nabla \cdot \mathbf{v} &= -cp, \end{aligned} \tag{1.1}$$

where c is a constant different from -1.

If $c = 0$, (1.1) describes the slow steady flow of a viscous fluid with velocity vector $\mathbf{v}$, pressure p, and external force $\mathbf{f}$ (where we have normalized the viscosity by setting it equal to one). A good reference for matters of this nature is [4, Chaps. 2 and 3].

For $c > 0$, (1.1) can be interpreted as the displacement equations governing a homogeneous isotropic elastic body in equilibrium, with $\mathbf{v}$ representing the infinitesimal displacement vector, $-cp$ the dilation, and $\mathbf{f}$ the body force suitably normalized (see [2, 5, (53.3)]).

The constant c in (1.1) is equal to $(1 - 2\sigma)$ where σ is Poisson's ratio, which expresses the fact that the shrinkage in a lateral direction due to stretching bears a constant ratio to the amount of stretch. To be specific, it is the ratio of the fractional lateral contraction to the fractional linear extension under tension. For most common substances, it is found that Poisson's ratio lies between 0 and $\frac{1}{2}$, e.g., $\sigma = 0.28$ for cast iron and $\sigma = 0.35$ for copper.

It is the purpose of this paper to study the discrete analog of the system of Eqs. (1.1) defined on the integral lattice points in 3-space (designated by M). The study of such type problems on lattice points in various Euclidean spaces appears to have attracted considerable attention through the years and has cut

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across many subject areas, i.e., probability theory, numerical and functional analysis, and partial differential equations (e.g., [1, 3, 6, 7, 9, 10]).

The main idea in this paper is to develop (evidently for the first time) the discrete analog of the fundamental solutions of Lorentz [8, pp. 110–111]. The asymptotic properties of said solutions are then used to establish the existence and the growth properties in a neighborhood of infinity of the solution to the discrete analog of (1.1). The case where the external force is in $L^{\gamma}(M)$, $3 \leqslant \gamma < \infty$, turns out to be the most interesting.

In the sequel, $m = (m_1, m_2, m_3)$ will designate an integral lattice point in E_3, M will be the set $\{m; m \text{ in } E_3\}$, and we shall study functions h defined on M. In particular, letting $e_j = (\delta_j^1, \delta_j^2, \delta_j^3)$ where δ_j^k is the Kronecker—δ, we set

$$\begin{aligned} D_j h(m) &= h(m + e_j) - h(m), \\ D_{-j} h(m) &= h(m) - h(m - e_j) \end{aligned} \tag{1.2}$$

for $j = 1, 2, 3$, and observe that

$$D_j[D_{-j}h(m)] = D_{-j}[D_j h(m)] = h(m + e_j) + h(m - e_j) - 2h(m). \tag{1.3}$$

$D_j h$ and $D_{-j} h$ are, respectively, the discrete analogs of the first derivative from the right and the left in the jth direction, and $D_j D_{-j} h$ is the discrete analog of the well-known second symmetric derivative in the jth direction.

Also, we shall set

$$Lh(m) = \sum_{j=1}^{3} D_j[D_{-j}h(m)] \tag{1.4}$$

and observe from (1.3) that L is the discrete analog of the Laplace operator.

With v_j, p, and f_j now representing functions defined on M, in this paper we intend to study the following system of equations.

$$\begin{aligned} Lv_j - D_j p &= -f_j, \qquad j = 1, 2, 3, \\ \sum_{k=1}^{3} D_{-k} v_k &= -cp, \end{aligned} \tag{1.5}$$

where c is a constant and $c \neq -1$.

It is clear that the finite difference system (1.5) is the discrete analog of the differential system (1.1).

When $\mathbf{f}$ is equal to zero, we shall refer to the system (1.5) as $(1.5)_0$, i.e.,

$$\begin{aligned} Lv_j - D_j p &= 0, \qquad j = 1, 2, 3, \\ \sum_{k=1}^{3} D_{-k} v_k &= -cp. \end{aligned} \tag*{$(1.5)_0$}$$

We shall call the pair $(\mathbf{v}, p)$ a linear solution of $(1.5)_0$ provided the following obtains: (i) $v_j(m)$ is a linear polynomial in the components of m, $j = 1, 2, 3$; (ii) $p(m)$ is a constant; (iii) $(\mathbf{v}, p)$ satisfies $(1.5)_0$.

We shall say $\mathbf{f}$ is in $L^\gamma(M)$, $1 \leqslant \gamma < \infty$ if $\Sigma_m |f_j(m)|^\gamma < \infty$ for $j = 1, 2, 3$. With $|m| = (m_1^2 + m_2^2 + m_3^2)^{1/2}$, and motivated somewhat by [1, p. 240], we intend to establish the following result.

THEOREM. *Suppose there exists a constant γ with $1 \leqslant \gamma < \infty$ such that $\mathbf{f}$ is in $L^\gamma(M)$. Then there is a pair $(\mathbf{v}, p)$ with the following properties:*

(i) $(\mathbf{v}, p)$ *satisfies the system* (1.5) *on* M;

(ii) $v_j = o(|m|^{2(1-\gamma^{-1})})$ *and* $p = o(|m|^{2(1-\gamma^{-1})})$ *as* $|m| \to \infty$, $j = 1, 2, 3$;

(iii) $(\mathbf{v}, p)$ *is unique up to a linear solution of* $(1.5)_0$.

In case $1 < \gamma \leqslant 2$, we infer from (ii) that in (iii), $(\mathbf{v}, p)$ is unique up to a constant solution of $(1.5)_0$, i.e., v_j is unique up to a constant for $j = 1, 2, 3$, and p is unique or unique up to a constant corresponding to the cases $c \neq 0$ or $c = 0$, respectively. If $\gamma = 1$, $(\mathbf{v}, p)$ is unique.

The proof of (iii) is easy, and we give it here. Suppose in particular that the pair $(\mathbf{v}^*, p^*)$ is another solution of (1.5) with $v^*_j = o(|m|^{2(1-\gamma^{-1})})$ and $p = o(|m|^{2(1-\gamma^{-1})})$ as $|m| \to \infty$. Then the pair $(\mathbf{v}^* - \mathbf{v}, p^* - p)$ is a solution of $(1.5)_0$, and we infer from this fact that

$$\sum_{j=1}^{3} D_{-j}[L(v^*_j - v_j) - D_j(p^*_j - p_j)] = 0.$$

Now, L and D_{-j} commute, and we conclude from (1.4) and this last fact that

$$-cL(p^* - p) - L(p^* - p) = 0.$$

But $(c + 1) \neq 0$, and we therefore have that $p^* - p$ is a discrete harmonic function, i.e., $L(p^* - p) = 0$.

Next, we use [3, Theorem 6] in conjunction with the fact that $p^* - p = o(|m|^2)$ as $|m| \to \infty$ and obtain that

$$p^*(m) - p(m) = a_{11}m_1 + a_{12}m_2 + a_{13}m_3 + a_0 . \tag{1.6}$$

From the first equation in (1.5), we consequently have that

$$L(v^*_j - v_j) = a_{1j} , \tag{1.7}$$

But then $L[v^*_j(m) - v_j(m) - a_{1j}m_j^2/2] = 0$. Therefore $v^*_j(m) - v_j(m) - a_{1j}m_j^2/2$ is a discrete harmonic function which is $O(|m|^2)$ as $|m| \to \infty$. Using [3, Theorem 6] once again, we obtain that $v^*_j(m) - v_j(m)$ is a polynomial of degree

2 in m_1, m_2, m_3. But $v^*_j(m) - v_j(m)$ is in particular $o(|m|^2)$. Consequently, in this polynomial, all the coefficients of the terms of degree 2 must vanish. We conclude first that $v^*_j(m) - v_j(m)$ is a linear polynomial and next, from (1.7), that $a_{1j} = 0$ for $j = 1, 2, 3$. But then from (1.6), we have that $p^*(m) - p(m)$ is equal to a constant. Therefore the pair $(\mathbf{v}^* - v, p^* - p)$ is a linear solution of $(1.5)_0$, and the proof of part (iii) of the theorem is complete.

2. Fundamental Lemmas

To establish parts (i) and (ii) of the theorem, we shall need some lemmas in the theory of three-dimensional Fourier series.

We shall set $T_3 = \{x, -\pi \leqslant x_j < \pi, j = 1, 2, 3\}$. So, in particular, if $\mathscr{F}$ is a Lebesgue integrable function on T_3 (the three-dimensional torus),

$$(2\pi)^{-3} \int_{T_3} e^{-i(m,x)} \mathscr{F}(x)\, dx$$

will designate the usual mth Fourier coefficient of $\mathscr{F}$.

Next, we shall set

$$\begin{aligned} \mathscr{P}_j(x) &= [1 - e^{i(e_j, x)}], \qquad j = 1, 2, 3 \\ &= [1 - e^{ix_j}] \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \mathscr{P}(x) &= \sum_{j=1}^{3} \mathscr{P}_j(x)\, \mathscr{P}_j(-x) \\ &= 6 - \sum_{j=1}^{3} (e^{ix_j} + e^{-ix_j}) \\ &= 4 \sum_{j=1}^{3} \sin^2\left(\frac{x_j}{2}\right). \end{aligned} \tag{2.2}$$

From (2.1), we observe that

$$\begin{aligned} D_j e^{i(m,x)} &= -\mathscr{P}_j(x)\, e^{i(m,x)}, \\ D_{-j} e^{i(m,x)} &= \mathscr{P}_j(-x)\, e^{i(m,x)}, \end{aligned} \tag{2.3}$$

and obtain as a consequence of (1.4) and (2.2) that

$$L e^{i(m,x)} = -\mathscr{P}(x)\, e^{i(m,x)}. \tag{2.4}$$

We next define the following functions for x in $T_3 - \{0\}$ and $j, k = 1, 2, 3$:

$$\mathscr{U}_j^k(x) = (1 + c)\, \delta_j^k / \mathscr{P}(x) - \mathscr{P}_j(-x)\, \mathscr{P}_k(x) / \mathscr{P}^2(x) \tag{2.5}$$

and

$$\mathcal{Q}^k(x) = -\mathcal{P}_k(x)/\mathcal{P}(x). \tag{2.6}$$

In (2.5), δ_j^k is the familiar Kronecker-δ and c is a constant different from -1.

Next, we observe that for x in $T_3 - \{0\}$,

$$\mathcal{P}(x)\,\mathcal{U}_j^k(x) - \mathcal{P}_j(-x)\,\mathcal{Q}^k(x) = (1 + c)\,\delta_j^k \tag{2.7}$$

and

$$\sum_{j=1}^{3} \mathcal{P}_j(x)\,\mathcal{U}_j^k(x) = -c\mathcal{Q}^k(x). \tag{2.8}$$

From (2.1), (2.2), and (2.6), we see that $\mathcal{U}_j^k(x)$ and $\mathcal{Q}^k(x)$ are in $L^1(T_3)$, and consequently we can define

$$u_j^k(m) = (1 + c)^{-1}\,(2\pi)^{-3} \int_{T_3} e^{-i(m,x)}\mathcal{U}_j^k(x)\,dx \tag{2.9}$$

and

$$q^k(m) = (1 + c)^{-1}\,(2\pi)^{-3} \int_{T_3} e^{-i(m,x)}\,\mathcal{Q}^k(x)\,dx. \tag{2.10}$$

We next establish the following lemma.

Lemma 1. *With $u_j^k(m)$ and $q^k(m)$ defined by* (2.9) *and* (2.10), *respectively, the following prevails:*

$$\begin{aligned} Lu_j^k(m) - D_j q^k(m) &= 0 &&\text{for } m \neq 0 \\ &= -\delta_j^k &&\text{for } m = 0; \end{aligned} \tag{2.11}$$

$$\sum_{s=1}^{3} D_{-s} u_s^k(m) = -cq^k(m). \tag{2.12}$$

From (2.3), (2.4), and (2.7), we obtain

$$\begin{aligned} &Lu_j^k(m) - D_j q^k(m) \\ &\quad = -(2\pi)^{-3} \int_{T_3} e^{-i(m,x)}[\mathcal{P}(-x)\,\mathcal{U}_j^k(x) - \mathcal{P}_j(-x)\,\mathcal{Q}^k(x)]\,(1 + c)^{-1}\,dx \\ &\quad = -(2\pi)^{-3} \int_{T_3} \delta_j^k e^{-i(m,x)}\,dx, \end{aligned}$$

and our assertion (2.11) is established.

To establish (2.12), we observe from (2.3) and (2.8) that

$$\sum_{j=1}^{3} D_{-j}u_j^k(m) = (2\pi)^{-3} \sum_{j=1}^{3} \int_{T_3} e^{-i(m,x)}(1 + c)^{-1} \mathscr{P}_j(x)\, \mathscr{U}_j^k(x)\, dx$$

$$= -c(2\pi)^{-3} \int_{T_3} e^{-i(m,x)}(1 + c)^{-1} \mathscr{Q}^k(x)\, dx.$$

Equation (2.12) follows immediately from this last computation and (2.10).

We next establish the following lemma:

LEMMA 2. *Suppose that* $\sum_m |f_j(m)| < \infty$ *for* $j = 1, 2, 3$. *Set* $v_j(m) = \sum_{k=1}^{3} \sum_n u_j^k(m - n) f_k(n)$ *and* $p(m) = \sum_{k=1}^{3} \sum_n q^k(m - n) f_k(m)$. *Then*

(i) *the pair* $(\mathbf{v}, p)$ *satisfies the system* (1.5);

(ii) $p(m) = o(1)$ *and* $v_j(m) = o(1)$ *as* $|m| \to \infty, j = 1, 2, 3$.

From (2.9) and (2.10), we have that both $u_j^k(m)$ and $q^k(m)$ are $o(1)$ as $|m| \to \infty$. Consequently $\sum_n |u_j^k(m - n) f_k(n)| < \infty$ and $\sum_n |q^k(m - n) f_k(n)| < \infty$. We therefore obtain from Lemma 1 that

$$Lv_j(m) - D_j q^k(m) = \sum_{k=1}^{3} \sum_n [Lu_j^k(m - n) - D_j q^k(m - n)] f_k(n)$$

$$= -\sum_{k=1}^{3} \delta_j^k f_k(m).$$

Also, from Lemma 1 we obtain

$$\sum_{j=1}^{3} D_{-j}v_j(m) = \sum_{j,k=1}^{3} \sum_n D_{-j}u_j^k(m - n) f_k(n)$$

$$= \sum_{k=1}^{3} \sum_n (-c)\, q^k(m - n) f_k(n).$$

The conclusion to part (i) of the lemma follows immediately from these last two computations. Part (ii) of the lemma follows easily from the Lebesgue dominated convergence theorem and the fact that u_j^k and q^k are uniformly bounded and $o(1)$ as $|m| \to \infty$.

Next, we establish the following lemma:

LEMMA 3. *There is a constant* A'_{jk} *such that*

$$\Big| (2\pi)^{-3} \int_{T_3} e^{-i(m,x)} \mathscr{P}_j(-x)\, \mathscr{P}_k(x)\, \mathscr{P}^{-2}(x)\, dx - (8\pi)^{-1} [\delta_j^k |m|^{-1} - m_j m_k |m|^{-3}]$$
$$- (16\pi)^{-1} |m|^{-5} [m_j |m|^2 - m_k |m|^2 - 3m_j m_k^2 + 3m_k m_j^2] \Big|$$
$$\leqslant A'_{jk} |m|^{-3}$$

for $|m| \geqslant 1$ *and* $j, k = 1, 2, 3$.

To establish Lemma 3, we proceed in a manner similar to that found in [1, pp. 238–240]. In particular, with $B(x, \rho)$ designating the open ball with center x and radius ρ, we see from this last reference that there is a function $\mathscr{F}_{jk}(x)$ such that the following hold:

$$\mathscr{F}_{jk}(x) \qquad \text{is in} \qquad C^{\infty}[B(0, 1) - \{0\}]; \tag{2.13i}$$

$$\mathscr{F}_{jk}(x) = O(|x|^2) \qquad \text{as} \qquad |x| \to 0; \tag{2.13ii}$$

the first, second, third, and fourth mixed partial derivatives of $\mathscr{F}_{jk}(x)$ are, respectively, $O(|x|)$, $O(1)$, $O(|x|^{-1})$, $O(|x|^{-2})$ as $|x| \to 0$; (2.13iii)

and

$$\begin{aligned} &\mathscr{P}_j(-x)\, \mathscr{P}_k(x)\, \mathscr{P}^{-2}(x) \\ &\quad = x_j x_k\, |x|^{-4} + i2^{-1}[x_j x_k^2 - x_j^2 x_k]\, |x|^{-4} \\ &\quad + 2\mathscr{L}_4(x)\, x_j x_k\, |x|^{-2} + b_{4jk}(x)\, |x|^{-4} \\ &\quad + i(x_j x_k^2 - x_k x_j^2)\, \mathscr{L}_4(x)\, |x|^{-2} + i b_{5jk}(x)\, |x|^{-4} + \mathscr{F}_{jk}(x) \end{aligned} \tag{2.14}$$

for x in $B(0, 1) - \{0\}$ where $b_{4jk}(x)$ and $b_{5jk}(x)$ are homogeneous polynomials in x_j, x_k of degrees 4 and 5, respectively, and $\mathscr{L}_4(x) = 2(x_1^4 + x_2^4 + x_3^4)\, |x|^{-4}/4!$

Next, with $0 < \rho_1 < \rho_2 < 1$, we choose a function $\mathscr{S}(\rho)$ such that

$$\begin{aligned} \mathscr{S}(\rho) &= 1 \qquad \text{for } 0 \leqslant \rho \leqslant \rho_1 \\ &= 0 \qquad \text{for } \rho_2 \leqslant \rho < \infty, \end{aligned} \tag{2.15}$$

and such that

$$\mathscr{S} \text{ is in class } C^{\infty} \text{ on the nonnegative real axis.} \tag{2.16}$$

Also, we set

$$\begin{aligned} \mathscr{G}_{jk}(x) &= \mathscr{P}_j(-x)\, \mathscr{P}_k(x)\, \mathscr{P}^{-2}(x) - \mathscr{S}(|x|)\, \mathscr{H}_{jk}(x) && \text{for } 0 < |x| < 1 \\ &= \mathscr{P}_j(-x)\, \mathscr{P}_k(x)\, \mathscr{P}^{-2}(x) && \text{for } x \text{ in } T_3 - B(0, 1) \\ &= 0 && \text{for } x = 0, \end{aligned} \tag{2.17}$$

where $\mathscr{H}_{jk}(x)$ designates the right-hand side of the equality in (2.14). It follows from (2.13),..., (2.17) that $\mathscr{G}_{jk}(x)$ has a periodic extension to all of E_3 such that $\mathscr{G}_{jk}$ is in $C^{\infty}(E_3)$. Consequently, we obtain, in particular, that

$$\int_{T_3} e^{-i(m,x)} \mathscr{G}_{jk}(x)\, dx = O(|m|^{-4}) \qquad \text{as} \qquad |m| \to \infty. \tag{2.18}$$

Next, we observe from (2.17) that

$$\int_{T_3} e^{-i(m,x)}[\mathscr{P}_j(-x)\,\mathscr{P}_k(x)\,\mathscr{P}^{-2}(x) - \mathscr{G}_{jk}(x)]\,dx$$
$$= \int_{B(0,1)} e^{-i(m,x)}\mathscr{S}(|\,x\,|)\,\mathscr{H}_{jk}(x)\,dx. \tag{2.19}$$

From (2.13) and [1, p. 239], it follows that

$$\int_{B(0,1)} e^{i(m,x)}\mathscr{S}(|\,x\,|)\,\mathscr{F}_{jk}(x)\,dx = O(|\,m\,|^{-4}). \tag{2.20}$$

From [1, Theorem 6], we next observe that

$$\begin{aligned}\int_{B(0,1)} & e^{-i(m,x)}\mathscr{S}(|\,x\,|)\,[2\mathscr{L}_4(x)\,x_jx_k\,|\,x\,|^{-2} + b_{4jk}(x)\,|\,x\,|^{-4} \\ & + i(x_jx_k^2 - x_kx_j^2)\,\mathscr{L}_4(x)\,|\,x\,|^{-2} + ib_{5jk}(x)\,|\,x\,|^{-4}]\,dx \\ & = O(|\,m\,|^{-3}).\end{aligned} \tag{2.21}$$

Also, from [1, Theorem 6], we see that there is a constant A''_{jk} such that

$$\begin{aligned}\Big|\,(2\pi)^{-3}\int_{T_3} & e^{-i(m,x)}\mathscr{S}(|\,x\,|)\,\{x_jx_k\,|\,x\,|^{-4} + i2^{-1}[x_jx_k^2 - x_j^2x_k]\,|\,x\,|^{-4}\}\,dx \\ & - (8\pi)^{-1}\,[\delta_j^k\,|\,m\,|^{-1} - m_jm_k\,|\,m\,|^{-3}] \\ & - (16\pi)^{-1}\,|\,m\,|^{-5}\,[m_j\,|\,m\,|^2 - m_k\,|\,m\,|^2 - 3m_jm_k^2 + 3m_km_j^2]\,\Big| \\ & \leqslant A''_{jk}\,|\,m\,|^{-3}\end{aligned} \tag{2.22}$$

for $|\,m\,| \geqslant 1$ and $j, k = 1, 2, 3$.

The conclusion to the lemma follows immediately from the definition of $\mathscr{H}_{jk}(x)$ and (2.18),..., (2.22).

Combining Lemma 3 with (2.5), (2.9), and [1, Theorem 2], we obtain the following lemma:

LEMMA 4. *There is a constant A_{jk} such that*

$$\begin{aligned}|\,u_j^k(m) - (4\pi)^{-1}\,\delta_j^k\,|\,m\,|^{-1} + (1+c)^{-1}\{(8\pi)^{-1}\,[\delta_j^k\,|\,m\,|^{-1} - m_jm_k\,|\,m\,|^{-3}] \\ - (16\pi)^{-1}\,|\,m\,|^{-5}\,[m_k\,|\,m\,|^2 - m_j\,|\,m\,|^2 + 3m_jm_k^2 - 3m_km_j^2]\}| \\ \leqslant A_{jk}\,|\,m\,|^{-3}\end{aligned}$$

for $|\,m\,| \geqslant 1$ and $j, k = 1, 2, 3$.

In a similar manner, we obtain the following lemma:

LEMMA 5. *There is a constant A_k such that*

$$| q^k(m) - [4\pi(1 + c)]^{-1} m_k | m |^{-3} | \leqslant A_k | m |^{-3}$$

for $| m | \geqslant 1$ *and* $k = 1, 2, 3$.

3. PROOF OF THE THEOREM

From the start because of Lemma 2, we can assume in this section that $1 < \gamma < \infty$. To prove part (i) of the theorem under this assumption, we set

$$f_j^k = f_j \delta_j^k, \qquad j, k = 1, 2, 3. \tag{3.1}$$

Next, we observe from the linearity of (1.5) that it is sufficient for the proof to find for each fixed k a pair $(\mathbf{v}^k, p^k)$ such that

$$\begin{aligned} Lv_j^k(m) - D_j p^k(m) &= -f_j^k(m), \qquad j = 1, 2, 3, \\ \sum_{s=1}^{3} D_{-s} v_s^k(m) &= -cp^k(m) \end{aligned} \tag{3.2}$$

for m in M.

With k fixed, we now establish (3.2). First of all, we notice from the linearity of (1.5) and from (2.11) and (2.12) that with no loss in generality, we can also suppose that

$$f_k^k(0) = 0. \tag{3.3}$$

Next, we make the following definitions:

$$\begin{aligned} 4\pi a_j^k(x) &= \delta_j^k | x |^{-1} - 2^{-1}(1 + c)^{-1} [\delta_j^k | x |^{-1} - x_j x_k | x |^{-3}] \\ &= \delta_j^k | x |^{-1} - 2^{-1}(1 + c)^{-1} \partial^2 | x | / \partial x_j \, \partial x_k , \end{aligned} \tag{3.4}$$

$$b_{js}^k(x) = \partial a_j^k(x) / \partial x_s , \tag{3.5}$$

$$(1 + c)\, 16\pi\, d_j^k(x) = | x |^{-5} [x_k | x |^2 - x_j | x |^2 + 3x_j x_k^2 - 3x_k x_j^2], \tag{3.6}$$

and finally

$$4\pi\alpha^k(x) = (1 + c)^{-1} x_k | x |^{-3} = -(1 + c)^{-1} \partial | x |^{-1} / \partial x_k \tag{3.7}$$

where in the above $x \neq 0$ and $j, s = 1, 2, 3$.

From (3.4), (3.6), and Lemma 4, we observe that for $| m | \geqslant 1$,

$$| u_j^k(m) - a_j^k(m) - d_j^k(m) | \leqslant A_{jk} | m |^{-3}. \tag{3.8}$$

Consequently, we have that

$$| u_j^k(m-n) - a_j^k(m-n) - d_j^k(m-n)| \leqslant 8A_{jk} | n |^{-3} \tag{3.9}$$

for $| n | \geqslant 2 | m | + 1$ and $j = 1, 2, 3$.

Similarly, we obtain from (3.7) and Lemma 5 that

$$| q^k(m-n) - \alpha^k(m-n)| \leqslant 8A_k | n |^{-3} \tag{3.10}$$

for $| n | \geqslant 2 | m | + 1$.

Next, using the fact that a_{jk} is an even function of x, we obtain from (3.5) and the mean-value theorem that there are constants B_{jk} such that

$$\left| a_j^k(m-n) - a_j^k(n) + \sum_{s=1}^{3} b_{js}^k(n)\, m_s \right| \leqslant B_{jk} | m |^2 | n |^{-3} \tag{3.11}$$

for $| n | \geqslant 2 | m | + 1$ and $j = 1, 2, 3$.

Likewise, we see there is a constant B_k such that

$$\begin{aligned} &| d_j^k(m-n) - d_j^k(-n)| \leqslant B_k | m | | n |^{-3}, \\ &| a_j^k(m-n) - a_j^k(n)| \leqslant B_k | m | | n |^{-2}, \\ \text{and} \quad &| \alpha^k(m-n) - \alpha^k(-n)| \leqslant B_k | m | | n |^{-3} \end{aligned} \tag{3.12}$$

for $| n | \geqslant 2 | m | + 1$ and $j = 1, 2, 3$.

Continuing with the proof of (3.2), where $f_j^k(m)$ is given by (3.1) and (3.3), $\mathbf{f}$ is in $L^\gamma(M)$, and $1 < \gamma < \infty$, we next apply Hölder's inequality and obtain that there is a constant ξ_k such that for $| m | \geqslant 1$,

$$\begin{aligned} &\sum_{|n|\geqslant 2|m|+1} | f_k^k(n)| | n |^{-3} < \xi_k | m |^{-3/\gamma} \qquad \text{for } 3 \leqslant \gamma < \infty, \\ &\sum_{|n|\geqslant 2|m|+1} | f_k^k(n)| | n |^{-2} < \xi_k | m |^{1-3/\gamma} \qquad \text{for } 1 < \gamma < 3. \end{aligned} \tag{3.13}$$

From (3.9), (3.10), (3.11), and (3.12), we see that there is a constant β_k with the property that

$$\left| u_j^k(m-n) - a_j^k(n) + \sum_{s=1}^{3} b_{js}^k(n)\, m_r - d_j^k(-n) \right| \leqslant \beta_k | m |^2 | n |^{-3} \tag{3.14}$$

and also that

$$\begin{aligned} &| u_j^k(m-n) - a_j^k(n) - d_j^k(-n)| \leqslant \beta_k | m | | n |^{-2} \\ \text{and} \quad &| q^k(m-n) - \alpha^k(-n)| \leqslant \beta_k | m | | n |^{-3} \end{aligned} \tag{3.15}$$

for $| n | \geqslant 2 | m | + 1$, $j = 1, 2, 3$, and $| m | \geqslant 1$.

Using (3.13), (3.14), and (3.15), we define

$$v_j^k(m) = \sum_{n\neq 0}\left[u_j^k(m-n) - a_j^k(n) + \sum_{s=1}^{3} b_{js}^k(n)\, m_s - d_j^k(-n)\right] f_k^k(n) \qquad \text{if} \quad 3 \leqslant \gamma < \infty$$

$$= \sum_{n\neq 0}\left[u_j^k(m-n) - a_j^k(n) - d_j(-n)\right] f_k^k(n) \qquad \text{if} \quad 1 < \gamma < 3, \tag{3.16}$$

and

$$p^k(m) = \sum_{n\neq 0}\left[q^k(m-n) - \alpha^k(-n)\right] f_k^k(n) \qquad \text{if} \quad 3 \leqslant \gamma < \infty$$

$$= \sum_{n\neq 0} q^k(m-n) f_k^k(n) \qquad \text{if} \quad 1 < \gamma < 3. \tag{3.17}$$

From (3.13), (3.14), and (3.15), we have that the series defined in (3.16) and (3.17) are absolutely convergent for every value of m. Consequently, the difference operators commute with the summation sign in each case, and we obtain, in particular, that

$$Lv_j^k(m) - D_j p^k(m) = \sum_n \left[Lu_j^k(m-n) - D_j q^k(m-n)\right] f_k^k(n).$$

We conclude from (2.11) and (3.1) that

$$Lv_j^k(m) - D_j p^k(m) = -\delta_j^k f_k^k(m) = -f_j^k(m) \tag{3.18}$$

for m in M.

Next, using (3.4), (3.5), and (3.7), we make a computation and observe that

$$\sum_{j=1}^{3} b_{jj}^k(n) = -c\alpha^k(n)$$

for $n \neq 0$.

As a consequence, we obtain that

$$\sum_{j=1}^{3} D_{-j}\left(\sum_{s=1}^{3} b_{js}^k(n)\, m_s\right) = \sum_{s=1}^{3} b_{ss}^k(n) = c\alpha^k(-n).$$

From this last fact in conjunction with (2.12), (3.16), and (3.17), we conclude that if $3 \leqslant \gamma < \infty$

$$\sum_{j=1}^{3} D_{-j} v_j^k(m) = -c \sum_{n\neq 0}\left[q^k(m-n) - \alpha^k(-n)\right] f_k^k(n)$$

$$= -cp^k(m)$$

for m in M. Likewise, if $1 < \gamma < 3$, $\sum_{j=1}^{3} D_{-j} v_j^k(m) = -cp^k(m)$.

This fact in conjunction with (3.18) gives (3.2) and the proof of part (i) of the theorem is complete.

To prove part (ii) of the theorem under the assumption $1 < \gamma < \infty$, we observe from Lemma 4 that $u_j^k(m) = O(|m|^{-1})$ and from Lemma 5 that $q^k(m) = O(|m|^{-2})$. Consequently, we see that in establishing part (ii), it is sufficient to show that for fixed k

$$v_j^k(m) = o(|m|^{2/\eta}) \qquad \text{as} \qquad |m| \to \infty, \quad j = 1, 2, 3, \tag{3.19i}$$

$$p^k(m) = o(|m|^{2/\eta}) \qquad \text{as} \qquad |m| \to \infty \tag{3.19ii}$$

where $v_j^k(m)$ is defined by (3.16), $p^k(m)$ by (3.17), and

$$\gamma^{-1} + \eta^{-1} = 1. \tag{3.20}$$

We also note for future reference that

$$\left\{\sum_{n\neq 0} |f_k^k(n)|^\gamma\right\}^{1/\gamma} < \infty. \tag{3.21}$$

Using the fact that the number of integral lattice points contained in the spherical annulus of inner radius R and outer $R + 1$ is $O(R^2)$ as $R \to \infty$, we apply Hölder's inequality and obtain

$$\sum_{|m|-1\leqslant|n|\leqslant|m|+1} |f_k^k(n)| = o(|m|^{2/\eta}) \qquad \text{as} \qquad |m| \to \infty. \tag{3.22}$$

Next, we see from Lemmas 4 and 5 and (3.4),..., (3.7) that there is a constant K such that

$$|u_j^k(m)| \leqslant K(|m| + 1)^{-1}, \tag{3.23}$$

$$|q^k(m)| \leqslant K(|m| + 1)^{-2}, \tag{3.24}$$

$$|a_j^k(m)| \leqslant K(|m| + 1)^{-1}, \tag{3.25}$$

$$|b_{js}^k(m)| \leqslant K(|m| + 1)^{-2}, \tag{3.26}$$

$$|d_j^k(m)| \leqslant K(|m| + 1)^{-2}, \tag{3.27}$$

and finally

$$|\alpha^k(m)| \leqslant K(|m| + 1)^{-2}, \tag{3.28}$$

where in the above $j, s = 1, 2, 3$.

We are now ready to establish (3.19i). To do so, we suppose first that $3 \leqslant \gamma < \infty$ and set

$$\left(\sum_{1\leqslant|n|<2|m|+1} + \sum_{|n|\geqslant 2|m|+1} \right) \times \left\{ \left[u_j^k(m-n) - a_j^k(n) + \sum_{s=1}^{3} b_{js}^k(n)\, m_s - d_j^k(-n) \right] f_k^k(n) \right\} = I_1(m) + I_2(m) \tag{3.29}$$

and observe from (3.16) that since $3 \leqslant \gamma < \infty$,

$$v_j^k(m) = I_1(m) + I_2(m). \tag{3.30}$$

From (3.14) and (3.29), we see that for $|m| \geqslant 1$,

$$|I_2(m)| \leqslant \beta_k |m|^2 \sum_{|n|\geqslant 2|m|+1} |n|^{-3} |f_k^k(n)| .$$

But using (3.13), we conclude in particular that

$$I_2(m) = o(|m|^{2/\gamma}) \qquad \text{as} \qquad |m| \to \infty. \tag{3.31}$$

From (3.23), (3.25), (3.26), (3.27), and (3.29), we see that

$$\begin{aligned} |I_1(m)| \leqslant K \sum_{1\leqslant|n|<2|m|+1} &[(|m-n|+1)^{-1} \\ &+ |n|^{-1} + 3|m||n|^{-2} + |n|^{-2}] |f_k^k(n)| . \end{aligned} \tag{3.32}$$

Now if $1 \leqslant |n| < 2|m| + 1$, then $|n|^{-1} \leqslant 3|m||n|^{-2}$. So we infer from (3.32) that

$$|I_1(m)| \leqslant 7K[I_{11}(m) + I_{12}(m)] \tag{3.33}$$

where

$$I_{11}(m) = \sum_{1\leqslant|n|<2|m|+1} (|m-n|+1)^{-1} |f_k^k(n)| \tag{3.34}$$

and

$$I_{12}(m) = |m| \sum_{1\leqslant|n|<2|m|+1} |n|^{-2} |f_k^k(n)| . \tag{3.35}$$

Applying Hölder's inequality to the sum on the right in (3.35), we see from (3.21) that $|I_{12}(m)|$ is majorized by a constant multiple of

$|m| \{\sum_{1\leqslant|n|\leqslant 2|m|} |n|^{-2\eta}\}^{1/\eta}$. But since $1 < \eta \leqslant \frac{3}{2}$, this in turn is majorized by a constant multiple of $|m|^{2/\eta} \log |m| \, |m|^{-1/\gamma}$. We conclude that

$$I_{12}(m) = o(|m|^{2/\eta}) \qquad \text{as} \qquad |m| \to \infty. \tag{3.36}$$

Next, we see from (3.22) that the sum on the right in (3.34) is majorized by

$$\left(\sum_{1\leqslant|n|\leqslant|m|-1} + \sum_{|m|+1\leqslant|n|<2|m|+1} \right) |m-n|^{-1} \, |f_k^k(n)| + o(|m|^{2/\eta}).$$

We consequently obtain from Hölder's inequality and the fact that $1 < \eta \leqslant \frac{3}{2}$ that

$$\begin{aligned} |I_{11}(m)| &\leqslant 2 \left\{ \sum_{1\leqslant|m|\leqslant 3|m|} |n|^{-\eta} \right\}^{1/\eta} \left\{ \sum_{n\neq 0} |f_k^k(n)|^{\gamma} \right\}^{1/\gamma} + o(|m|^{2/\eta}) \\ &\leqslant O(|m|^{2/\eta} \, |m|^{-1/\gamma}) + o(|m|^{2/\eta}). \end{aligned}$$

Consequently

$$|I_{11}(m)| = o(|m|^{2/\eta}) \qquad \text{as} \qquad |m| \to \infty. \tag{3.37}$$

This fact in conjunction with (3.36) and (3.33) gives us that $I_1(m) = o(|m|^{2/\eta})$ as $|m| \to \infty$. But in turn this fact in conjunction with (3.30) and (3.31) gives us (3.19i) for the case $3 \leqslant \gamma < \infty$.

To establish (3.19i) for the case $1 < \gamma < 3$, we retrace our steps to (3.29) and (3.16) and see that we have to show that

$$I'_1(m) = \sum_{1\leqslant|n|<2|m|+1} |u_j^k(m-n) - a_j^k(n) - d_j^k(-n)| \, |f_k^k(n)|$$

and

$$I'_2(m) = \sum_{|n|\geqslant 2|m|+1} |u_j^k(m-n) - a_j^k(n) - d_j^k(-n)| \, |f_k^k(n)|$$

are both $o(|m|^{2/\eta})$ as $|m| \to \infty$. From (3.13) and (3.15) we see with no difficulty that $I'_2(m) = o(|m|^{2/\eta})$. To handle $I'_1(m)$, we observe that

$$|I'_1(m)| \leqslant 2K \sum_{1\leqslant|n|<2|m|+1} [(|m-n|+1)^{-1} + |n|^{-1}] \, |f_k^k(n)| \, .$$

But $\sum_{1\leqslant|n|<2|m|+1} |n|^{-1} \, |f_k^k(n)| = o(|m|^{2/\eta})$, and an argument similar to that used for (3.34) shows that $I'_1(m)$ is indeed $o(|m|^{2/\eta})$. The proof of (3.19i) is therefore complete.

It remains to establish (3.19ii). To do this, we use (3.17), assume $3 \leqslant \gamma < \infty$, and write $p^k(m)$ as two sums as in (3.29) and (3.30) obtaining

$$p^k(m) = II_1(m) + II_2(m). \tag{3.38}$$

We then use the second inequality in (3.15), obtain as before that $| II_2(m)| \leqslant | m | \beta_k \sum_{|n|\geqslant 2|m|+1} | n |^{-3} | f_k{}^k(n)|$, and conclude from (3.13) that

$$II_2(m) = o(| m |^{2/\eta}) \qquad \text{as} \qquad | m | \to \infty. \tag{3.39}$$

Next, we see from (3.17), (3.24), and (3.28) that

$$| II_1(m)| \leqslant K \sum_{1\leqslant|n|<2|m|+1} [(| m - n | + 1)^{-2} + | n |^{-2}] \, | f_k{}^k(n)| .$$

But it follows from this fact, (3.34), and (3.35) that $| II_1(m)| \leqslant K | I_{11}(m)| + K | I_{12}(m)|$. Using (3.36) and (3.37), we consequently obtain that $II_1(m) = o(| m |^{2/\eta})$ as $| m | \to \infty$, and (3.19ii) is established for the case $3 \leqslant \gamma < \infty$.

The case $1 < \gamma < 3$ entails an analysis of $\sum_{n\neq 0} (| m - n | + 1)^{-2} | f_k{}^k(n)|$ which, using the methods just presented, is easily seen to be $o(| m |^{2/\eta})$ as $| m | \to \infty$. Consequently, (3.19ii) is established in all cases, and therefore part (ii) of the theorem is established. Since we have already shown earlier that part (iii) of the theorem holds, the proof of the theorem is complete.

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